

Non-vanishing elements in finite groups

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Abstract

Many results have been established about determining whether or not an element evaluates to zero on an irreducible character of a group. In this note it is shown that if a group G has a normal nilpotent subgroup N , and P is a Sylow p -subgroup of G , then no irreducible character of G vanishes on $N \cap Z(P)$.

Let G be a finite group and $\chi \in \text{Irr}(G)$, an irreducible character of G . A classical result of Burnside says if χ is non-linear, that is $\chi(1) \neq 1$, then there is at least one element g in G such that $\chi(g) = 0$. If one considers conjugacy classes, a natural dual to irreducible characters, then g being a central element in G implies that $|\chi(g)| = \chi(1)$ and thus g does not evaluate to zero on any irreducible character. However, a non-central element g may also not evaluate to zero on any irreducible character, for example the 3-cycles in $\text{Sym}(3)$. Elements which do not evaluate to zero on any irreducible character of a group are called non-vanishing. The study of non-vanishing elements was first introduced in [INW99], where the authors showed for soluble groups any non-vanishing element g in a group G must reduce to a 2-element in $G/F(G)$. In [DNP⁺10] this result was generalised to any group, in particular it was shown that if an element g is non-vanishing in G and the order of g is coprime to 6, then g lies in $F(G)$.

We note that there has been a recent interest in the literature asking about how much group structure is determined by the vanishing conjugacy class sizes. In particular, in [DPS10] and [Bro], the authors have generalised arithmetical results upon conjugacy classes to vanishing conjugacy classes. Thus the determination of non-vanishing elements would provide further machinery for this recent topic of research.

The aim of this note is to generalise one of the key results in [INW99], that is [INW99, Theorem A], which says if a group has a normal Sylow p -subgroup P , then all the elements in $Z(P)$ are non-vanishing. A variant of this result was considered in [Miy12], where the author showed that if a group has a normal elementary abelian p -subgroup A and P is a Sylow p -subgroup, then the elements in $Z(P) \cap A$ are non-vanishing. In particular, we first show that the result of [Miy12] holds if A is a normal abelian subgroup. From this we deduce the result holds if A is a normal nilpotent subgroup. Note that from this new result, [INW99, Theorem A] follows by setting $A = P$.

Theorem. *Let G be a finite group, which contains a non-trivial normal nilpotent subgroup N and p a prime. Then for $P \in \text{Syl}_p(G)$, the elements in $N \cap Z(P)$ are non-vanishing in G .*

First we give the following preliminary result which considers when a sum of roots of unity is equal to zero.

Lemma. Let $\Xi := \{\xi_1, \dots, \xi_t\}$ be a set of p^n -th roots of unity, for some number $n \geq 1$, such that $\xi_1 + \dots + \xi_t = 0$. Then the sum can be split into sums of the form $\xi + \xi^{p^{a-1}+1} + \dots + \xi^{(p-1)p^{a-1}+1}$, for possibly various numbers $1 \leq a \leq n$, where $\xi^{kp^{a-1}+1} \in \Xi$ for $0 \leq k \leq p-1$ and each such subsum equals zero.

Proof. Let ξ be an element in $\{\xi_i \mid 1 \leq i \leq t\}$ of maximal order, i.e. ξ is a primitive p^a -th root of unity and $\xi_i^{p^a} = 1$ for all i . It is enough to prove that $\xi, \xi^{p^{a-1}+1}, \dots, \xi^{(p-1)p^{a-1}+1} \in \Xi$, as then

$$\xi + \xi^{p^{a-1}+1} + \dots + \xi^{(p-1)p^{a-1}+1} = \xi(1 + \xi^{p^{a-1}} + \dots + \xi^{(p-1)p^{a-1}}) = 0$$

and inductively from $\Xi \setminus \{\xi, \xi^{p^{a-1}+1}, \dots, \xi^{(p-1)p^{a-1}+1}\}$ repeat the argument.

Assume $\xi_1 = \xi$, which is a primitive p^a -th root of unity, so that each ξ_i is a power of ξ . Pick r minimal such that $\sum_{j=1}^r \xi^{b_j} = 0$ with $\xi^{b_j} \in \Xi$, where $b_i \leq b_{i+1} \leq p^a$ and $b_1 = 1$. Then it follows that ξ is a root to the polynomial $\sum_{j=1}^r X^{b_j}$. As $\Phi_{p^a}(X) = 1 + X^{p^{a-1}} + \dots + X^{(p-1)p^{a-1}}$ is the minimal polynomial for ξ it follows that

$$\sum_{j=1}^r X^{b_j} = \Phi_{p^a}(X)g(X),$$

for some polynomial $g \in \mathbb{Z}[X]$.

For each j , $b_j > 0$, therefore $g(X)$ cannot have a constant term, i.e. $g(X) = Xf(X)$ for some $f \in \mathbb{Z}[X]$. The polynomial $f(X)$ must have a constant term $c \neq 0$, because $b_1 = 1$. Thus

$$\sum_{j=1}^r X^{b_j} = cX\Phi_{p^a}(X) + X^2\Phi_{p^a}(X)h(X),$$

for some $h \in \mathbb{Z}[X]$. As $b_j \leq p^a$, it follows that $2 + (p-1)p^{a-1} + \deg(h(X)) \leq p^a$. Moreover, $2 \leq \deg(X^2h(X)) \leq p^{a-1}$. If $X^2\Phi_{p^a}(X)h(X)$ has a term of the form $X^{kp^{a-1}+1}$ then $X^2h(X)$ must have the term X or $X^{mp^{a-1}+1}$ for some positive integer m , which is a contradiction. In particular, $cX\Phi_{p^a}(X)$ has no terms in common with $X^2\Phi_{p^a}(X)h(X)$. Hence $X\Phi_{p^a}(X)$ occurs as a subsum of $\sum_{j=1}^r X^{b_j}$. However as $\xi\Phi_{p^a}(\xi) = 0$ it follows that

$$\sum_{j=1}^r X^{b_j} = X\Phi_{p^a}(X).$$

□

We can now establish the main result in the case that a group has a normal abelian subgroup. Note that the proof makes use of the method in [Miy12] with the additional information about roots of unity in the previous lemma.

Proposition. Let G be a finite group, which contains a non-trivial normal abelian subgroup A and p a prime. Then for $P \in \text{Syl}_p(G)$, the elements in $A \cap Z(P)$ are non-vanishing in G .

Proof. Let $x \in A \cap Z(P)$ such that there exists some $\chi \in \text{Irr}(G)$ for which $\chi(x) = 0$. By Clifford's theorem $\chi \downarrow_A = e \sum_{i=1}^t \zeta^{g_i}$ such that $\zeta \in \text{Irr}(A)$ and the set $\{g_i\}$ forms a transversal of $I := I_G(\zeta)$ in G , for $I_G(\zeta)$ the inertial subgroup in G of ζ . If $\chi \downarrow_A(x) = 0$, then $\sum_{i=1}^t \zeta^{g_i}(x) = 0$. Thus by the lemma we can split this sum into smaller subsums of p elements (which also equal zero). Let $\{\xi_j\}$ denote a set of representatives for the distinct subsums of p elements as in the above lemma. If k_j denotes the multiplicity of ξ_j , then $\sum_j k_j p = t$. Hence it is enough to show that the p -part of t , denoted t_p , divides k_j as then $t_p p$ divides t which is a contradiction.

The fact that the multiplicity k_j is divisible by t_p is in the proof of [Miy12, Theorem], however we shall include details for completeness.

The subgroup P acts on the set of G -conjugates of ζ with the orbit size of ζ^g given by

$$|P : P \cap I^g| = |G : P \cap I^g|_p = |G : I^g|_p |I^g : P \cap I^g|_p.$$

Therefore $|G : I|_p$ divides the orbit size. As $x \in Z(P)$ it follows that $\zeta^{gy}(x) = \zeta^g(x)$ for all $y \in P$. Thus the value x evaluates to is constant on each P -orbit. Hence the multiplicity of ξ in $\{\zeta^{g_i}(x)\}$ must be divisible by $|G : I|_p = t_p$.

This completes the proof. \square

From the proposition the main theorem now follows.

Proof of theorem. Let N be a non-trivial normal nilpotent subgroup of a finite group G and P a Sylow p -subgroup of G for a prime p . Assume $x \in N \cap Z(P)$. As N is nilpotent, $O_p(N)$ is the unique Sylow p -subgroup of N . Thus $x \in O_p(N) \cap Z(P)$. As $O_p(N)$ is a normal p -subgroup of G , the group $O_p(N)$ is a subgroup of P . In particular, it follows that $O_p(N) \cap Z(P) \leq Z(O_p(N))$. Hence $x \in Z(O_p(N)) \cap Z(P)$ and the result follows from the proposition, as $Z(O_p(N))$ is a normal abelian subgroup of G . \square

Corollary (1). *Let G be a finite group and $F(G)$ the fitting subgroup of G . Then for p a prime and $P \in \text{Syl}_p(G)$, the elements in $F(G) \cap Z(P)$ are non-vanishing in G .*

In [INW99] it is conjectured that for soluble groups, all non-vanishing elements lie in the Fitting group. This result provides a partial insight into which elements of the Fitting subgroup would in fact be non-vanishing in a group.

In the proof of the proposition, the importance of A being abelian is that the restricted character must be a sum of linear characters. It is therefore natural to assume that if such a restriction was made to characters with degree not divisible by p , a similar argument should work. In fact this can be bypassed in a more general setting by using the Ito-Michler Theorem, which was proven for soluble groups by N. Ito [It51] and then for any group by G. Michler [Mic86] using the classification of finite simple groups.

Corollary (2). *Let G be a finite group and N normal in G such that no irreducible character of N has degree divisible by a prime p . Furthermore, let P be a Sylow p -subgroup of G . Then the elements in $Z(P) \cap N$ are non-vanishing in G .*

Proof. As N has no irreducible characters of degree divisible by p , the Ito-Michler theorem implies that N has a normal Sylow p -subgroup Q which is abelian. Therefore $N \cap Z(P) = Q \cap Z(P)$. Moreover, as Q is a normal Sylow p -subgroup of N , Q is normal in G . Hence Q is a normal abelian subgroup of G . Thus by the theorem, the elements of $Q \cap Z(P)$ are non-vanishing in G . \square

Finally note that in the above corollary the condition that no irreducible character has degree divisible by p cannot be removed. In particular, let G be a non-abelian simple group of Lie type with order divisible by p . Then G has an irreducible character of r -defect zero for any prime r and hence any non-trivial element in G is vanishing [Isa94, Theorem 8.17]. Thus for $G = N$ the conclusion of the corollary cannot hold.

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